

# From quantum spectra to classical orbits: Varying rectangular billiards

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## Abstract

The quantum-spectrum function is introduced by means of the eigenvalues and the eigenfunctions in the system of varying rectangular billiards. It is found that the Fourier transform of the quantum-spectrum function corresponds with the classical orbits in the system. These results give new evidence about the classical-quantum correspondence.

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## 1. Introduction

The characteristic of the classical theory is that there exist objects, phenomena, and events that are distinct and well-defined and events that exhibit reliable and reproducible properties with the aid of which they can be identified and compared. However, when we come to describe quantum concepts, we find that the precision of our customary scientific language leads to difficult and unwieldy modes of expression. As is well known, the quantum properties of matter are to be associated with incompletely defined potentialities, which can be more definitely realized only in interaction with a classically describable system.

In fact, instead of having well-defined variables that are in a one-to-one correspondence with the actual behavior of matter, we have at the quantum level a wave function that is only in statistical correspondence with this behavior. Nevertheless, it is only at the classical level that definite results for an experiment can be obtained in the form of distinct events, which are associated in a one-to-one correspondence with the various possible values of the physical

quantity that is measured. This means that quantum theory presupposes the classical level and the general correctness of classical concepts in describing this level. It does not deduce the classical concepts as limiting cases of quantum concept [1]. Thus, the correspondence principle is simply a consistency condition which requires that when the quantum theory plus its classical interpretation is carried to the limit of high quantum numbers, the simple classical theory will be obtained. Hence, classical definiteness and quantum potentialities complement each other in providing a complete description of the system as a whole.

Quantum theory has actually evolved in such a way that it implies the need for a new concept or the relation between large-scale and small-scale properties of a given system. In 1971, Gutzwiller introduced the periodic-orbit theory of states density [2,3]. This theory has been proven to be a powerful tool in its original quest of understanding the role of periodic orbits in quantum systems, where the classical motion is chaotic. Gutzwiller's theory revealed a kind of new relationship between macroscopic and microcosmic properties of a system.

From that time on, studies on dynamical behavior of classical and quantum mechanics are continual for the integrable and chaos systems [3,4]. The research results have

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become indispensable tools for understanding some new physical phenomena. For example, Du and Delos [5–7] have recognized that the periodic-orbit theory is the right theoretical tool for atomic spectra. They found that it was not periodic orbits, but close orbits that produce the visible signal in the absorption spectrum, and the “quasi-Landau resonance” phenomenon can be explained successfully by this theory. Thus, the form of periodic-orbit theory that gives a description of atomic spectra is known as closed-orbit theory.

The closed-orbit theory has been extended and refined in a number of ways, and used to study a variety of phenomena [8–11]. We can determine the periods and the classical actions of orbits of the electron, and consider the creation and bifurcation of these orbits as the dynamics of the system changing from orderly to chaotic. The interpretation of atomic spectra in terms of classical orbits has been called “recurrence spectroscopy”. Recently, the theory has been reformulated to consider the effects of time-dependent fields on the electron [12]. There have also been interesting recent proposals for retrieving long-time quantum information from short-time classical orbits [13].

In this study, we will propose a kind of new classical-quantum correspondence principle. We will define a new quantum-spectrum function using the eigenvalues and the eigenfunctions of a system. And as an instance, we will calculate the quantum-spectrum function in the system of varying rectangular billiards. The correspondence relationship between the quantum-spectrum function and the classical orbits will be shown. These orbits are neither periodic orbits of Gutzwiller nor closed orbits of Du and Delos.

## 2. New quantum-spectrum function

We consider a multi-dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} + V(q) \quad (1)$$

If  $A(x_1, y_1)$  and  $B(x_2, y_2)$  are two arbitrary fixed points in the system, we define the new quantum-spectrum function as

$$\rho_{AB}(E) \equiv \sum_n \psi_n^*(A) \psi_n(B) \delta(E - E_n) \quad (2)$$

where  $n$  is the quantum number,  $E_n$  is the energy eigenvalue, and  $\psi_n$  is the corresponding eigenfunction.

In analogy with the reckon of closed-orbit theory,  $\rho_{AB}(E)$  can always be written as

$$\rho_{AB}(E) = \rho_{AB}^0(E) + \sum_i C_i \sin(S_i + \varphi_i) \quad (3)$$

It is apparent that there are a background term and a summation of many sine resonance terms in Eq. (3). The summation includes all the classical reasonable orbits from point A to point B. Here, the amplitude  $C_i$  is related to the stability of orbits, the action variable  $S_i = \int_A^B p \, dq$  is the

integral along the orbits, and  $\varphi_i$  is the phase, including the Maslov phase corrections. It should be pointed out that Eq. (3) shows the relationship between the quantum-spectrum function [Eq. (2)] and the classical orbits.

## 3. Quantum spectra in varying rectangular billiards

As a simple explicit example of the calculation of quantum-spectrum function, we consider the two-dimensional motion of a billiard with unit mass in the varying rectangular, with length  $\lambda a$  and width  $\lambda b$ , where  $\lambda$  is a parameter of scale, which can vary from zero to infinity continuously. The billiard is restrained by reflecting walls that terminate a region of constant potential energy. The potential function for the system can be expressed as

$$V(x, y) = \begin{cases} 0, & 0 < x < \lambda a, \quad 0 < y < \lambda b \\ \infty, & \text{Otherwise} \end{cases} \quad (4)$$

The corresponding stationary state Schrödinger equation is

$$-\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x, y) = E \psi(x, y) \quad (5)$$

This equation can be solved by separation of variables. The energy eigenvalues of Eq. (5) can be given by

$$E_{mn} = \frac{\pi^2}{2\lambda^2} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right), \quad m, n = 1, 2, 3, \dots \quad (6)$$

and the corresponding eigenfunctions are

$$\psi_{mn}(x, y) = \frac{1}{\lambda} \sqrt{\frac{2}{ab}} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \quad (7)$$

If we set that  $A(\lambda x_1, \lambda y_1)$  and  $B(\lambda x_2, \lambda y_2)$  are two arbitrary points in the varying rectangular (not on the boundary), we can write the new quantum-spectrum function as

$$\rho_{AB}(E, \lambda) = \sum_{mn} \psi_{mn}^*(\lambda x_1, \lambda y_1) \psi_{mn}(\lambda x_2, \lambda y_2) \delta(E - E_{mn}) \quad (8)$$

Now we will prove that the new quantum-spectrum function Eq. (8) contains the information of classical orbits from point A to point B by means of the Fourier transformation. Choosing a fixed energy  $E$ , and considering the characteristic of the motion for a billiard in the varying rectangular [14,15], we can write the action variable as  $S = kL$ , where  $k = \sqrt{2E}$  is the absolute value of momentum and  $L$  is the length of orbit. It can be seen that the amplitude  $C_i$  varies with  $1/\sqrt{L}$  for the two-dimensional system, so we should multiply  $\sqrt{\lambda}$  with the Fourier integral. Thus, we can define the Fourier transformation as

$$\tilde{\rho}_{AB}(L, S) \equiv \int_0^{+\infty} d\lambda \sqrt{\lambda} \rho_{AB}(E, \lambda) \exp(i\lambda S) \quad (9)$$

Substituting Eq. (8) into Eq. (9) and integrating Eq. (9), we can obtain

$$\tilde{\rho}_{AB}(E, S) = \sum_{mn} \frac{1}{ab} E^{-\frac{1}{4}} E^{-\frac{3}{4}} \sin \frac{m\pi x_1}{a} \sin \frac{m\pi y_1}{b} \times \sin \frac{m\pi x_2}{a} \sin \frac{m\pi y_2}{b} \exp \left( i \sqrt{\frac{E_{mn}}{E}} S \right) \quad (10)$$

The summation in Eq. (10) includes all the energy levels between  $E_{\min}$  and  $E_{\max}$ .

From Eq. (3), considering that  $C_i$  changes slowly in the integral regions and can be regarded as constants approximately, through the same Fourier transformation we obtain

$$\tilde{\rho}_{AB}(E, S) \approx \tilde{\rho}_{AB}^0(E, S) + \sum_i C_i \left[ -\frac{\exp(-i\varphi_i)}{2i} \right] \Delta(S - S_i) \quad (11)$$

where

$$\tilde{\rho}_{AB}^0(E, S) = \int_0^{+\infty} \rho_{AB}^0(E, \lambda) \exp(i\lambda S) d\lambda \quad (12)$$

is the background term, and

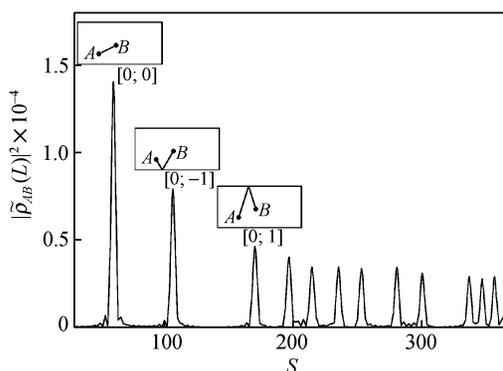


Fig. 1. Norm of Fourier transformation of quantum spectra  $|\tilde{\rho}_{AB}(E, S)|^2$  changing with the self-variable  $S$ . The forms of the corresponding classical orbits are drawn on the first three peaks.

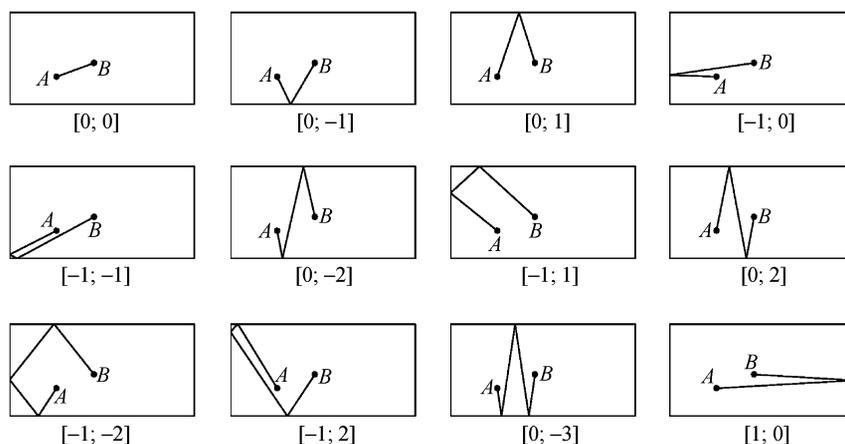


Fig. 2. Classical orbits from point A to point B in the rectangular with length of less than 13. The classical orbits correspond with the peaks in Fig. 1 one-by-one.

$$\Delta(x) \equiv \int_0^{+\infty} \exp(i\lambda x) d\lambda \quad (13)$$

is a function with its peak at the point of  $x = 0$ , and the revolving wave approximation is used.

From Eq. (11), we can find that the positions of peaks of function  $|\tilde{\rho}_{AB}(E, S)|^2$  will correspond with the length of the classical orbits from point A to point B.

#### 4. Quantum spectra and classical orbits

Losing no generality, as an illustration, we choose the following parameters

$$\begin{aligned} a &= 9.96, & b &= 4.56, \\ x_1 &= 2.52, & y_1 &= 1.23, \\ x_2 &= 4.54, & y_2 &= 2.06, \\ E &= 381.26 \end{aligned}$$

to calculate the function. All the states with energy eigenvalue lower than  $E_{0\max} = 800$  are calculated in the summation. Fig. 1 shows the situation that the norm of Fourier transformation of quantum spectra  $|\tilde{\rho}_{AB}(E, S)|^2$  changes with the self-variable  $S$ .

The billiard does the rectilinear motion in the rectangular till it collides with the walls of rectangular and is reflected. We have found all the classical orbits from point A to point B with length less than 13 (Fig. 2). We will depict the forms of every orbit by the method as follows:

- (i) All of the orbits are expressed by two integers in square brackets. A semicolon separates the two integers.
- (ii) The absolute value of the first integer means the number of times that the orbit collides with the two walls vertical to  $x$ -axis, and the absolute value of the second integer means the number of times that the orbit collides with the two walls vertical to  $y$ -axis.

Table 1  
 Contrast about positions of quantum peaks obtained by calculation and nature of classical orbits in rectangular billiards (from point A to point B)

Number	Positions of quantum peaks	Forms of classical orbits	Length of classical orbits ( $L$ )	Classical action variable ( $S = kL$ )
1	60.3	[0;0]	2.18	60.3
2	106.6	[0;-1]	3.86	106.6
3	170.3	[0;+1]	6.17	170.4
4	196.8	[-1;0]	7.11	196.3
5	214.7	[-1;-1]	7.79	215.1
6	235.3	[0;-2]	8.53	235.6
7	253.2	[-1;+1]	9.16	252.8
8	280.6	[0;+2]	10.15	280.4
9	300.4	[-1;-2]	10.89	300.7
10	336.9	[-1;+2]	12.20	336.9
11	346.9	[0;-3]	12.57	347.2
12	356.6	[+1;0]	12.89	355.9

- (iii) The sign of the first integer means the direction of  $x$ -axis when the billiard leaves from point A. The “+” means in the same direction as  $x$ -axis, and the “-” means in the reverse direction of  $x$ -axis. The sign of the second integer means the direction of  $y$ -axis when the billiard leaves from point A. The “+” means in the same direction as  $y$ -axis, and the “-” means in the reverse direction of  $y$ -axis.

The last three columns in Table 1 give the nature of classical orbits. It is easy to find that all the classical orbits of the billiard correspond one-to-one with the positions of the quantum peaks in Fig. 1 (the norm of Fourier transformation of quantum spectra). From Table 1, we can also find that the action variable  $S$  of each classical orbit is just consistent with the position of each quantum peak within the range of allowable error.

## 5. Conclusions

We have introduced the quantum spectrum by using the eigenvalues and the eigenfunctions in the system of varying rectangular billiards. This research has showed that, in the system of varying rectangular billiards, there is the quantitative correspondence relation between the classical orbits and the quantum spectra defined in this paper. We also find that, these orbits are the opened orbits from point A to point B, and points A and B can be chosen arbitrarily. Although the discussion about classical-quantum corre-

spondence is only in the system of varying rectangular billiards, we believe that all the methods and results can be used in many other systems, including some one-dimensional and three-dimensional systems.

On the other hand, the dynamics of all kind of rectangular billiards is closely related to the transport nature of nano-device. Hence, research about these systems is very important in the field of applied science.

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